On the 5D differential calculus and translation transformations in 4D κ-Minkowski noncommutative spacetime

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Abstract

We perform a Noether analysis for a description of translation transformations in 4D κ-Minkowski noncommutative spacetime which is based on the structure of a 5D differential calculus. The techniques that some of us had previously developed (hep-th/0607221) for a description of translation transformations based on a 4D differential calculus turn out to be applicable without any modification, and they allow us to show that the basis usually adopted for the 5D calculus does not take into account certain aspects of the structure of time translations in κ-Minkowski. We propose a change of basis for the 5D calculus which leads to a more intuitive description of time translations.

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I. INTRODUCTION AND SUMMARY

A rather sizeable literature has been devoted over these past few years (see, *e.g.*, Refs. [1, 2, 3] and references therein) to the possibility that the short-distance (Planck-scale) structure of spacetime, which according to a popular "quantum-gravity intuition" [4, 5, 6, 7, 8] should be highly nontrivial, might be such to require a new description of spacetime symmetries. In particular, the quantum-gravity version [9] of Minkowski spacetime, which should be like classical Minkowski spacetime for soft probes but should reveal additional structures when probed with sensitivities approaching the Planck length L_p ($L_p = \sqrt{\hbar G/c^3} \sim 10^{-35}m$), might require some deformation of the Poincaré symmetries. Unfortunately, several grey areas remain in the formalization of this intriguing hypothesis. A nonclassical spacetime which several authors have considered as a candidate for the emergence of modified spacetime symmetries is the κ-Minkowski noncommutative spacetime [10, 11], with the characteristic noncommutativity

$$[x_j, x_0] = i\lambda x_j \tag{1}$$

$$[x_k, x_j] = 0 \ , \tag{2}$$

where x_0 is the time coordinate, x_j are space coordinates ($j, k \in \{1, 2, 3\}$), and λ is a length scale, usually expected to be of the order of the Planck length.

κ-Minkowski is considered a promising candidate because it admits a "natural" formulation [10, 12, 13] of rules of transformation under translations, space-rotations and boosts, and the generators of these transformations can be described as generators of the so-called "κ-Poincaré" Hopf algebra [10, 11, 14] rather than a standard Lie algebra. The Hopf-algebra structures intervene primarily in the law of action of generators on products of functions, since the noncommutativity of the coordinates turns out to be incompatible with the Lie-algebra requirement $T(fg) = T(f)g + fT(g)$ (if f and g are function of the κ-Minkowski coordinates and T is one of the generators of the transformations).

While it is easy to show that "κ-Poincaré" structures can be naturally used to describe some mathematical features of field theories in the κ-Minkowski spacetime, it has been frustratingly difficult to establish if and how these structures affect the physics of these theories, in spite of more than a decade of studies by several research groups. Until not long ago one could still rightfully argue (as done, *e.g.*, in Ref. [15]) that the Hopf-algebra structures might provide only a fancy mathematical formalization of a rather trivial break down of symmetry. However, some of us recently reported [16] a first successful Noether analysis for some "κ-Poincaré" symmetries of κ-Minkowski, and clearly the conserved charges derived in Ref. [16], even though we only managed to obtain them for the translation sector (and we did not provide a "measurement theory" for them), represent a significant step forward in the direction of establishing that indeed the κ-Poincaré structures encountered in the description of field theories in the κ-Minkowski spacetime really amount to a manifestation of a short-distance deformation of physical/observable symmetries.

The key ingredient of the Noether analysis reported in Ref. [16] is the introduction of translation-transformation parameters with appropriate nontrivial commutators with the spacetime coordinates. In looking for a suitable description of translation transformations in 4D κ-Minkowski spacetime, Ref. [16] naturally assumed that 4 such parameters would be needed, and indeed the Noether analysis was successful adopting a description of the action of translation transformations on functions of the type $f \to f + df$, with ($\mu \in \{0, 1, 2, 3\}$)

$$df = i\epsilon_\mu P^\mu f(x) \ , \tag{3}$$

where the operators P^μ are the previously known "generators of the Majid-Ruegg basis of κ-Poincaré" [10, 11], while for the four "noncommutative transformation parameters" ϵ_μ it turns out [16] that one must introduce product rules with the spacetime coordinates that are modelled on the structure of a 4D differential calculus known for κ-Minkowski [17], amounting to the requirements $[\epsilon_j, x_0] = i\lambda\epsilon_j$, $[\epsilon_j, x_k] = 0$, $[\epsilon_0, x_\mu] = 0$.

More recently, in Ref. [18], it was argued that it should also be possible to perform a Noether analysis of translation transformations in 4D κ-Minkowski using a description of translations inspired by the structure of a known 5D differential calculus for κ-Minkowski [19]. Indeed, the

possibility of a 5D differential calculus for a 4D spacetime, which one would not usually consider, is rather well established in the κ-Minkowski literature. While in classical Minkowski spacetime one has only one natural differential calculus, there is more then one consistent differential calculus for κ-Minkowski, and, besides some 4D differential calculi, the structure of κ-Minkowski is also compatible with the introduction of a certain 5D differential calculus [19], the so-called "bicovariant differential calculus".

In light of these considerations the argument put forward in Ref. [18] is rather reasonable. Still for a better understanding of this possibility of using a 5D differential calculus for a 4D spacetime it may be useful to perform a more in-depth analysis then the one reported in Ref. [18]. Moreover, since these are our first experiences with the workings of the Noether theorem when applied to Hopf-algebra spacetime symmetries, it is intriguing that in comparison with the previous Ref. [16] the analysis reported in Ref. [18] chooses a significantly different set up for the Noether analysis. Are these difference dictated by the structure of the 5D differential calculus? or could one analyze the case with the 5D calculus using the same setup that proved successful in Ref. [16]?

In particular, Ref. [18] relied on a proposed equivalence between a free κ-Minkowski field theory and a free relativistically invariant (non-local) field theory on classical Minkowski space-time. And in setting up this commutative-theory description a carefully taylored Weyl map was adopted. These represent significant differences with respect to the previous example of Noether analysis reported in Ref. [16], which only relied on direct explicit manipulations of noncommutative fields and did not appear to prefer any particular choice of Weyl map.

We here expose several issues that invite further scrutiny of the equivalence of theories proposed in Ref. [18], and in particular we observe that this proposed equivalence associates time translations of the commutative theory to transformations that are not time translations in the noncommutative theory. We have therefore chosen not to rely on attempts of reformulation in terms of a commutative theory, but proceed as in Ref. [16], by working directly in terms of the noncommutative fields. We find that the same tools developed in Ref. [16] for the 4D-calculus analysis are also well suited for dealing with the 5D-calculus analysis.

Our motivation goes beyond establishing the viability of the odd possibility of 5-parameter translation transformations in a 4D spacetime. We are mostly hoping to contribute to the understanding of spacetime symmetries in noncommutative spacetimes, and particularly κ-Minkowski. The fact that there are different (and not clearly equivalent) ways to describe the translation transformations of a given noncommutative spacetime is certainly very significant for the task of finding the correct physical/operative meaning of these symmetry transformations. By working explicitly in terms of the fields of interest, functions of the noncommutative κ-Minkowski spacetime coordinates, we manage to uncover several structures that were beyond the reach of the commutative-theory reformulation adopted in Ref. [18], and could be very significant for a full characterization of our 5-parameter transformations. In particular we find that within the 5D-calculus setup some subtleties must be handled when trying to establish the time independence of a noncommutative field and at present it is not possible to formulate even a tentative proposal of identification of the energy observable. More material for a debate on the physical interpretation of this framework is provided by the fact that we find 5 conserved charges from the Noether analysis of our 5-parameter transformations.

In the next section we set up the description of our 5 parameter transformations. Then in Section III we perform the Noether analysis, we show how time derivatives are to be formulated in the 5D-calculus setup, and obtain 5 time-independent charges. While the analysis in Section III implicitly assumes the fields to be real, in Section IV we provide generalized formulas applicable to complex fields. In Section V we propose a change of basis for the 5D calculus which reflects our findings on the description of time derivatives. In Section VI we comment on the fate of classical symmetries within our framework. In Section VII we compare our findings with the ones reported in Ref. [20], by the same authors of Ref. [18], which appeared while we were in the final stages of preparation of this manuscript. On several issues Ref. [20] adopts a perspective which is somewhat different from the one adopted in Ref. [18], but still there are very significant differences with respect to the analysis we are reporting in this manuscript. Section VIII offers some closing remarks.

II. 5-PARAMETER TRANSLATIONS FOR 4D κ-MINKOWSKI

The argument put forward in Ref. [18] is centered on a well-known peculiarity of κ-Minkowski, *i.e.* the availability of a "natural"[1] 5D differential calculus $\{\hat{d}x_0, \hat{d}x_j, \hat{d}x_4\}$ defined by the following commutation relations[2]

$$[x_0, \hat{d}x_4] = i\lambda\hat{d}x_0 \ , \quad [x_0, \hat{d}x_0] = i\lambda\hat{d}x_4 \ , \quad [x_0, \hat{d}x_j] = 0$$

$$[x_j, \hat{d}x_4] = [x_j, \hat{d}x_0] = -i\lambda\hat{d}x_j \ , \quad [x_j, \hat{d}x_k] = i\lambda\delta_{jk}(\hat{d}x_4 - \hat{d}x_0) \ . \tag{4}$$

Assuming that the translation-transformation parameters $\{\hat{\epsilon}_0, \hat{\epsilon}_j, \hat{\epsilon}_4\}$ are the elements of the 5D differential calculus ($\hat{\epsilon}_0 \equiv \hat{d}x_0$, $\hat{\epsilon}_j \equiv \hat{d}x_j$, $\hat{\epsilon}_4 \equiv \hat{d}x_4$) one can introduce infinitesimal translations of fields as maps $\Phi \to \Phi + \hat{d}\Phi$ with

$$\hat{d}\Phi = i\,(\,\hat{\epsilon}^0 \hat{P}_0 + \hat{\epsilon}^j \hat{P}_j + \hat{\epsilon}^4 \hat{P}_4\,)\,\Phi \tag{5}$$

where the operators $\hat{P}_0, \hat{P}_j, \hat{P}_4$ are simply related to the operators P_0, P_j used by some of us in the 4-parameter description of translations reported in Ref. [16]:

$$\begin{aligned}
\hat{P}_0 &= \tfrac{1}{\lambda}(\sinh\lambda P_0 + \tfrac{\lambda^2}{2}\vec{P}^2 e^{\lambda P_0}) \\
\hat{P}_i &= P_i e^{\lambda P_0} \\
\hat{P}_4 &= \tfrac{1}{\lambda}(\cosh\lambda P_0 - 1 - \tfrac{\lambda^2}{2}\vec{P}^2 e^{\lambda P_0})
\end{aligned} \tag{6}$$

The P_0, P_j used in Ref. [16] are the translation generators of the Majid-Ruegg basis of the κ-Poincaré Hopf algebra [10, 11], while the $\hat{P}_0, \hat{P}_j$ of (5) clearly provide a different basis of generators for κ-Poincaré. It is easy to verify [10, 13] that the fifth operator, $\hat{P}_4$, is a casimir of the κ-Poincaré Hopf-algebra [10, 11].

Since one can write any κ-Minkowski field $\Phi(x)$ in the form [13, 21]

$$\Phi(x) = \int d^4p\,\tilde{\Phi}(p)\,e^{i\vec{p}\cdot\vec{x}}\,e^{-ip_0x_0} \ , \tag{7}$$

in order to provide an explicit description of the action of the Majid-Ruegg generators it is sufficient to specify that

$$P_\mu \triangleright \left(e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0}\right) = k_\mu\,e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0}$$

and accordingly for the $\hat{P}_A$ operators one finds

$$\begin{aligned}
\hat{P}_0 \triangleright \left(e^{i\vec{k}\cdot\vec{x}}\,e^{-i\vec{k}_0x_0}\right) &= \frac{1}{\lambda}(\sinh\lambda k_0 + \frac{\lambda^2}{2}\vec{k}^2 e^{\lambda k_0})\,e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0} \ , \\
\hat{P}_i \triangleright \left(e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0\,x_0}\right) &= k_i\,e^{\lambda k_0}\;e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0} \ , \\
\hat{P}_4 \triangleright \left(e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0}\right) &= \frac{1}{\lambda}(\cosh\lambda k_0 - 1 - \frac{\lambda^2}{2}\vec{k}^2 e^{\lambda k_0})\,e^{i\vec{k}\cdot\vec{x}}\,e^{-ik_0x_0} \ .
\end{aligned} \tag{8}$$

[1] This 5D differential calculus is natural in the sense that it can be singled out by a criterion [19] of "bicovariance", which essentially amounts to demanding that the commutators between elements of the differential calculus and spacetime coordinates be covariant under the action of the generators of the κ-Poincaré Hopf algebra.

[2] We use capital latin letters for indices running over {0, 1, 2, 3, 4}, small latin letters for indices running over {1, 2, 3}, and greek letters for indices running over {0, 1, 2, 3}.

In light of the form of these rules of action we should explicitly warn the reader of the fact that our notation is potentially misleading: usually a 0 index on a P generator identifies a time-translation generator, but clearly our $\hat{P}_0$ does not generate time translations (in particular, it does not vanish on time-independent functions). We shall take this into account (actually the formalism itself will direct us toward the correct way to handle this issue) and comment again on this point when appropriate.

In preparation for our Noether analysis it is useful to observe that (using the rules (8) of action of the $\hat{P}_A$ operators and the rules of commutation between transformation parameters and κ-Minkowski coordinates) one easily verifies [13, 22, 23, 24] that the differential $\hat{d}\Phi$ defined in (5) satisfies Leibniz rule

$$\hat{d}(\Phi\Psi) = \Phi(\hat{d}\Psi) + (\hat{d}\Phi)\Psi\,. \tag{9}$$

For the Noether analysis it is also useful to notice that from the rules of commutation between transformation parameters (elements of the 5D differential calculus) and κ-Minkowski coordinates one obtains the following rules of commutation between transformation parameters and time-to-the-right-ordered exponentials:

$$\begin{aligned} e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\,\hat{\epsilon}_0 &= \left((\lambda\hat{P}_0 + e^{-\lambda P_0})\hat{\epsilon}_0 + \lambda\hat{P}_i\hat{\epsilon}_i + (\lambda\hat{P}_4 + 1 - e^{-\lambda P_0})\hat{\epsilon}_4\right) e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0} \\ e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\,\hat{\epsilon}_i &= \left(\lambda e^{-\lambda P_0}\hat{P}_i\hat{\epsilon}_0 + \hat{\epsilon}_i - \lambda e^{-\lambda P_0}\hat{P}_i\hat{\epsilon}_4\right) e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0} \\ e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\,\hat{\epsilon}_4 &= \left(\lambda\hat{P}_0\hat{\epsilon}_0 + \lambda\hat{P}_i\hat{\epsilon}_i + (\lambda\hat{P}_4 + 1)\hat{\epsilon}_4\right) e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\,. \end{aligned} \tag{10}$$

And it is useful to write down the "coproducts" of the operators $\hat{P}_A$,[3]

$$\Delta(\hat{P}_0) = \hat{P}_0 \otimes e^{\lambda P_0} + e^{-\lambda P_0} \otimes \hat{P}_0 + \lambda e^{-\lambda P_0}\hat{P}_i \otimes \hat{P}_i\,, \tag{11}$$

$$\Delta(\hat{P}_i) = \hat{P}_i \otimes e^{\lambda P_0} + 1 \otimes \hat{P}_i\,, \tag{12}$$

$$\Delta(\hat{P}_4) = \hat{P}_4 \otimes e^{\lambda P_0} - e^{-\lambda P_0} \otimes \hat{P}_0 - \lambda e^{-\lambda P_0}\hat{P}_i \otimes \hat{P}_i + 1 \otimes (\hat{P}_0 + \hat{P}_4)\,, \tag{13}$$

which describe in standard Hopf-algebra notation the action of the operators $\hat{P}_A$ on products of functions. For example, Eq. (12) reflects the fact that from (8) it follows that

$$\begin{aligned} \hat{P}_i\left(e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\cdot e^{i\vec{p}\cdot\vec{x}}e^{-ip_0x_0}\right) &= \hat{P}_i\left(e^{i(\vec{k}+e^{-\lambda k_0}\vec{p})\cdot\vec{x}}e^{-i(k_0+p_0)x_0}\right) = \\ &= (k_i + e^{-\lambda k_0}p_i)\,e^{\lambda(k_0+p_0)}\left(e^{i(\vec{k}+e^{-\lambda k_0}\vec{p})\cdot\vec{x}}e^{-i(k_0+p_0)x_0}\right) = \\ &= (k_i e^{\lambda(k_0+p_0)} + e^{\lambda p_0}p_i)\left(e^{i(\vec{k}+e^{-\lambda k_0}\vec{p})\cdot\vec{x}}e^{-i(k_0+p_0)x_0}\right) = \\ &= \hat{P}_i\left(e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\right)\cdot e^{\lambda P_0}\left(e^{i\vec{p}\cdot\vec{x}}e^{-ip_0x_0}\right) + \left(e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\right)\cdot\hat{P}_i\left(e^{i\vec{p}\cdot\vec{x}}e^{-ip_0x_0}\right)\,. \end{aligned}$$

[3] In writing these coproducts we found convenient, in order to keep equations short, to let intervene the operator P_0. In order to write these coproducts exclusively in terms of the operators $\hat{P}_A$ one can of course use the fact that from (6) it follows that $\exp(\lambda P_0) = \lambda(\hat{P}_0 + \hat{P}_4) + 1$.

III. NOETHER ANALYSIS

We are now ready to test our 5-parameter transformations within a Noether analysis for classical fields in the noncommutative κ-Minkowski spacetime. As mentioned in the Introduction it is interesting for us to verify whether this analysis can be performed following exactly the techniques already developed in Ref. [16] or it is necessary to adapt the procedure to some peculiarities of the 5D differential calculus. We shall therefore proceed exactly as in Ref. [16] and deduce from the success of this Noether analysis that it is not necessary to adapt the Noether analysis to the type of differential calculus in use. As basis for our illustrative example of Noether analysis we consider one of the most studied equations of motion in the κ-Minkowski literature [11, 13] which concerns a scalar field $\Phi(x)$ governed by the Klein-Gordon-like equation of motion

$$C_\lambda(P_\mu)\,\Phi \equiv \left[\left(\frac{2}{\lambda}\sinh\frac{\lambda}{2}P_0\right)^2 - e^{\lambda P_0}\vec{P}^2\right]\Phi = m^2\Phi \ , \qquad (14)$$

whose most general solution can be written in the form[4]

$$\Phi(x) = \int d^4k\,\tilde{\Phi}(k_0,\vec{k})\,e^{i\vec{k}\cdot\vec{x}}e^{-ik_0x_0}\,\delta(C_\lambda(k_\mu)-m^2). \qquad (15)$$

Our Noether analysis relies on the fact that this equation of motion can be derived [16] from the following action

$$\begin{aligned} S[\Phi] &= \int d^4x\mathcal{L}[\Phi(x)] \\ \mathcal{L}[\Phi(x)] &= \frac{1}{2}\big(\Phi(x)\,C_\lambda\,\Phi(x) - m^2\Phi(x)\Phi(x)\big) \end{aligned} \qquad (16)$$

The operator $C_\lambda(P_\mu)$ defined in (14) commutes with all the operators introduced in the previous section, since it is a Casimir of the κ-Poincaré Hopf algebra. We find sometimes useful to also write[5] it as $C_\lambda = \tilde{P}_\mu\tilde{P}^\mu$ in terms of the operators

$$\tilde{P}_0 = \frac{2}{\lambda}\sinh\frac{\lambda}{2}P_0\,, \qquad \tilde{P}_i = P_i e^{\frac{\lambda}{2}P_0}\,, \qquad (17)$$

which turn out to allow to write more compactly some of the equations.

We are of course interested in analyzing the variation of the Lagrangian density under our 5-parameter transformation. This can be done easily using Eqs. (5) and (10), noting that

$$\tilde{P_\alpha}[f(x)g(x)] = [\tilde{P_\alpha}f(x)][e^{\frac{\lambda}{2}P_0}g(x)] + [e^{-\frac{\lambda}{2}P_0}f(x)][\tilde{P_\alpha}g(x)]\,, \qquad (18)$$

and using the fact that, by definition of a scalar field (and assuming $\delta\Phi(x') \equiv \Phi'(x') - \Phi(x') \simeq \Phi'(x) - \Phi(x) \equiv \delta\Phi(x)$)

$$0 = \Phi'(x') - \Phi(x) = [\Phi'(x') - \Phi(x')] - [\Phi(x') - \Phi(x)]\ , \qquad (19)$$

i.e. $\delta\Phi = -\hat{d}\Phi = -i\left(\hat{\epsilon}^0\hat{P}_0 + \hat{\epsilon}^j\hat{P}_j + \hat{\epsilon}^4\hat{P}_4\right)\Phi$.

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[4] We consider the function C_λ, defined in (14), sometimes on operators, as in the case of $C_\lambda(P_\mu)$ which is an operator, and sometimes on Fourier parameters, as in the case of $C_\lambda(k_\mu)$ which is just a number obtained from the Fourier parameters.

[5] As customary, we adopt the Einstein summation rule for greek and latin indices.

The result (whose detailed derivation will be reported elsewhere [22, 23]) is

$$0 = \delta\mathcal{L} = \frac{1}{2}\left(\delta\Phi\, C_\lambda \Phi + \Phi\, C_\lambda\, \delta\Phi - m^2 \delta\Phi\, \Phi - m^2 \Phi\, \delta\Phi\right) =$$

$$= -\frac{1}{2}\left\{ e^{\frac{\lambda P_0}{2}} \tilde{P}^0 \left[\left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \Phi\, \delta\Phi - \Phi \frac{e^{-\lambda P_0}}{\lambda} \delta\Phi \right] + \hat{P}^i \left[\Phi e^{-\lambda P_0} \hat{P}_i \delta\Phi - \hat{P}_i \Phi\, \delta\Phi \right] \right\}, \quad (20)$$

where we already specialized to the case of fields such that $\tilde{P}^\mu \tilde{P}_\mu \Phi = m^2 \Phi$, since of course we perform the Noether analysis on fields that are solutions of the equation of motion.

In (20) the transformation parameters $\hat{\epsilon}_A$ appear implicitly through $\delta\Phi$. It is convenient to use the formulas (10) to carry all the $\hat{\epsilon}_A$ to the left side of the monomials composing the expression of $\delta\mathcal{L}$. This allows to rewrite Eq. (20) in the form [22, 23]

$$\hat{\epsilon}^A \left(e^{\frac{\lambda P_0}{2}} \tilde{P}^0\, J_{0A} + \hat{P}^i J_{iA} \right) = 0\,, \quad (21)$$

where

$$\begin{aligned}
J_{00} &= \frac{1}{2}\Bigg\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[(\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi + \lambda P_i \Phi \hat{P}_i \Phi + \lambda \hat{P}_0 \Phi \hat{P}_4 \Phi \right] + \\
&\quad - (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi - \lambda P_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - \lambda \hat{P}_0 \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \Bigg\}, \\
J_{0i} &= \frac{1}{2}\Bigg\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[\lambda \hat{P}_i \Phi \hat{P}_0 \Phi + \Phi \hat{P}_i \Phi + \lambda \hat{P}_i \Phi \hat{P}_4 \Phi \right] + \\
&\quad - \lambda \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi \hat{P}_0 \Phi - \Phi \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi - \lambda \hat{P}_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \Bigg\}, \\
J_{04} &= \frac{1}{2}\Bigg\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[(\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi - \lambda P_i \Phi \hat{P}_i \Phi + (\lambda \hat{P}_4 + 1) \Phi \hat{P}_4 \Phi \right] + \\
&\quad - (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \lambda P_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - (\lambda \hat{P}_4 + 1) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \Bigg\}. \quad (22)
\end{aligned}$$

First let us notice that the Noether analysis leads automatically to a structure that does not require terms of the type $\hat{P}^4 J_{4A}$, which is a reassuring feature considering the peculiar nature of the operator $\hat{P}^4$ needed for our 5-parameter transformations. It is even more noteworthy that the Noether analysis leads automatically to a structure of the form $\exp(\lambda P_0/2)\tilde{P}^0\, J_{0A} + \hat{P}^i J_{iA}$. The observations we reported in Section II concerning the operator $\hat{P}^0$ imply that it would have been puzzling if the Noether analysis had led to a structure of the type $\hat{P}^\mu J_{\mu A}$, since $\hat{P}^0$ does not vanish on time-independent fields. The Eq. (21) produced by the Noether analysis is at least plausible, since, in light of the definitions given in Section II, $\exp(\lambda P_0/2)\tilde{P}_0$ does vanish on time-independent fields. This motivates us to adopt the suggestive notation

$$\mathcal{D}_0 \equiv \exp(\lambda P_0/2)\tilde{P}_0\ . \quad (23)$$

But the evidence of robustness of Eq. (21) goes even beyond $\mathcal{D}_0$: we find that the role played by the structure $\mathcal{D}^0 J_{0A} + \hat{P}^i J_{iA}$ in our Noether analysis is completely analogous to the role of the 4-divergence of the currents in the Noether analysis of ordinary theories in classical Minkowski spacetime. In order to provide support for this statement we describe spatial integration in κ-Minkowski as codified in the formula

$$\int d^3x e^{i\vec{p}\cdot\vec{x}} e^{-ip_0\, x_0} = \delta(\vec{p})\, e^{-ip_0\, x_0}\,, \quad (24)$$

so that for a κ-Minkowski field $\Psi(x) = \int d^4p\, \tilde{\Psi}(p_0, \vec{p}) \exp(i\vec{p}\cdot\vec{x}) \exp(-ip_0\, x_0)$ one obtains

$$\int d^3x \Psi(x_0, \vec{x}) = \int dp_0 \tilde{\Psi}(p_0, \vec{0})\, e^{-ip_0\, x_0} \,. \tag{25}$$

With these rules of integration one easily obtains from (21) (which must be valid for any arbitrary choice of the parameters $\hat{\epsilon}^A$) the following chain of relations [22, 23]:

$$\mathcal{D}_0 \int d^3x\, J_{0A} = \int d^3x\, \mathcal{D}_0\, J_{0A} = -\int d^3x \hat{P}^i J_{iA} = 0 \,, \tag{26}$$

where on the right-hand side we used (24) and the rule of action of the operators $\hat{P}^i$ introduced in Section II.

This argument guarantees that the charges

$$\hat{Q}_A \equiv \int d^3x\, J_{0A} \tag{27}$$

are indeed time independent. But of course one does not need to rely on this argument, since the time independence of the charges can be verified directly. This explicit verification can be done rather straightforwardly using the techniques of Ref. [16]. By direct evaluation of the charges one obtains a result (whose derivation will be reported in detail elsewhere [22, 23]) which is indeed explicitly time independent, and can be most conveniently expressed in terms of the Fourier transform $\tilde{\Phi}(k)$ of a field $\Phi(x)$ solution of the equation of motion:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = -\frac{1}{2}\int d^4k\, \tilde{\Phi}(k)\tilde{\Phi}(\dot{-}k)\, e^{3\lambda k_0} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2|}\, \delta(C_\lambda(k) - m^2) \,, \tag{28}$$

where, for compactness, we introduced the notations $k \equiv (k_0, \vec{k})$, $\dot{-}k \equiv (-k_0, -\vec{k}e^{\lambda k_0})$, and (analogously to the notation $\tilde{P}_\mu$ previously introduced for frequently occurring combinations of the P_μ generators) $\tilde{k}|_{k_0,\vec{k}} \equiv \{\tilde{k}_0, \tilde{\vec{k}}\}|_{k_0,\vec{k}} \equiv \left\{\frac{2}{\lambda}\sinh(\frac{\lambda}{2}k_0), \vec{k}\exp(\frac{\lambda}{2}k_0)\right\}$, as well as $\{\hat{k}_0,\, \hat{k}_i,\, \hat{k}_4\}|_{k_0,\vec{k}} \equiv \left\{\frac{1}{\lambda}(\sinh\lambda k_0 + \frac{\lambda^2}{2}\vec{k}^2 e^{\lambda k_0}),\, k_i e^{\lambda k_0},\, \frac{1}{\lambda}(\cosh\lambda k_0 - 1 - \frac{\lambda^2}{2}\vec{k}^2 e^{\lambda k_0})\right\}$.

Besides being time independent, for real fields (and complex fields, but this we discuss in the next section) our charges are also automatically real, as one can verify using the fact that a scalar field $\Phi(x)$ solution of our equation of motion $C_\lambda(P)\,\Phi = m^2\Phi$ on κ-Minkowski,

$$\Phi(x) = \int d^4k\, \tilde{\Phi}(k)\, \delta(C_\lambda(k) - m^2) e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\, x_0} \,,$$

will be real if

$$\tilde{\Phi}(k_0, \vec{k}) = \left(\tilde{\Phi}(-k_0, -\vec{k}e^{\lambda k_0})\right)^* e^{3\lambda k_0} \,, \tag{29}$$

and this allows us to rewrite the charges (28) as explicitly real[6] quantities:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = -\frac{1}{2}\int d^4k \left|\tilde{\Phi}(k)\right|^2 \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2|}\, \delta(C_\lambda(k) - m^2) \,. \tag{30}$$

[6] We shall later contemplate the possibility of on-shell fields with complex k_0 Fourier parameter. Note, however, that $\hat{k}_0$, $\hat{k}_i$, $\hat{k}_4$ and $\tilde{k}_0 e^{\frac{\lambda}{2}k_0}$ are real even when k_0 has an imaginary part.

IV. THE CASE OF COMPLEX SCALAR FIELDS

In the previous section we considered real scalar fields, but actually the steps of the analysis only require little or no adaptation for the case of complex fields. Essentially it reduces to the fact that in appropriate places one must consider the complex conjugate $\Phi^*(x)$ of the field $\Phi(x)$. For a complex field (which really amounts to 2 real fields) most authors would also remove the factor 1/2 in front of the action, besides introducing the characteristic structure "$\Phi^* \dots \Phi$":

$$\begin{aligned} S[\Phi] &= \int d^4x \mathcal{L}[\Phi(x)] \\ \mathcal{L}[\Phi(x)] &= \left(\Phi^*(x)\, C_\lambda\, \Phi(x) - m^2 \Phi^*(x)\Phi(x)\right) . \end{aligned} \tag{31}$$

Proceeding exactly in the same way as in the previous section one then easily arrives once again to the equation $\hat{\epsilon}^A \left(e^{\frac{\lambda P_0}{2}} \tilde{P}^0\, J_{0A} + \hat{P}^i J_{iA} \right) = 0$, with J's of the same form as in the previous section but involving $\Phi^*(x)$ in appropriate places. In particular, one finds

$$\begin{aligned} J_{00} &= \left\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[(\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi^* \hat{P}_0 \Phi + \lambda P_i \Phi^* \hat{P}_i \Phi + \lambda \hat{P}_0 \Phi^* \hat{P}_4 \Phi \right] + \right. \\ &\quad \left. - (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi - \lambda P_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - \lambda \hat{P}_0 \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\} , \\ J_{0i} &= \left\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[\lambda \hat{P}_i \Phi^* \hat{P}_0 \Phi + \Phi^* \hat{P}_i \Phi + \lambda \hat{P}_i \Phi^* \hat{P}_4 \Phi \right] + \right. \\ &\quad \left. - \lambda \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi^* \hat{P}_0 \Phi - \Phi^* \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi - \lambda \hat{P}_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\} , \\ J_{04} &= \left\{ \left(\frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[(\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi^* \hat{P}_0 \Phi - \lambda P_i \Phi^* \hat{P}_i \Phi + (\lambda \hat{P}_4 + 1) \Phi^* \hat{P}_4 \Phi \right] + \right. \\ &\quad \left. - (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \lambda P_i \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - (\lambda \hat{P}_4 + 1) \Phi^* \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\} . \end{aligned} \tag{32}$$

And from these one obtains the time-independent charges, which are also conveniently written in terms of Fourier transforms:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = - \int d^4k\, \tilde{\Phi}(k) \tilde{\Phi}^*(\dot{-}k)\, e^{3\lambda k_0} \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{| - 2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|}\, \delta(C_\lambda(k) - m^2) . \tag{33}$$

Also for complex fields the charges are real, as one easily verifies using the relation [22, 23] between $\tilde{\Phi}$ and $\tilde{\Phi}^*$,

$$\tilde{\Phi}(k_0, \vec{k}) = \left(\tilde{\Phi}^*(-k_0, -\vec{k} e^{\lambda k_0}) \right)^* e^{3\lambda k_0} , \tag{34}$$

which allows to rewrite the charges in the following way:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = - \int d^4k \left| \tilde{\Phi}(k) \right|^2 \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{| - 2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|}\, \delta(C_\lambda(k) - m^2) \tag{35}$$

V. A DIFFERENT CHOICE OF BASIS FOR THE 5D CALCULUS

In this section we propose a change of basis for the 5D differential calculus that reflects our findings concerning the operator $\hat{P}_0$, which clearly could not be used to generate time translations, and our operator $\mathcal{D}_0 = \hat{P}_0 + \hat{P}_4$ which instead appears to be a good candidate as generator of time translations. In order to describe translation transformations explicitly in terms of $\mathcal{D}_0$ (rather than separately $\hat{P}_0$ and $\hat{P}_4$) we propose[7] the following change of basis $\hat{d}x_A \to \bar{d}x_A$:

$$\bar{d}x_0 = (\hat{d}x_0 + \hat{d}x_4)/\sqrt{2}, \qquad \bar{d}x_i = \hat{d}x_i, \qquad \bar{d}x_4 = (\hat{d}x_0 - \hat{d}x_4)/\sqrt{2}\,. \tag{36}$$

The rules of commutation between the $\bar{d}x_A$ and the κ-Minkowski coordinates, which one easily obtains from the corresponding rules of commutation between the $\hat{d}x_A$ and the κ-Minkowski coordinates, take a rather simple form:[8]

$$[x_0, \bar{d}x_0] = i\lambda \bar{d}x_0, \qquad [x_0, \bar{d}x_4] = -i\lambda \bar{d}x_4, \qquad [x_0, \bar{d}x_j] = 0,$$

$$[x_j, \bar{d}x_4] = 0, \qquad [x_j, \bar{d}x_0] = -\sqrt{2}\, i\lambda \bar{d}x_j, \qquad [x_j, \bar{d}x_k] = -\sqrt{2}\, i\lambda \delta_{jk} \bar{d}x_4\,. \tag{37}$$

Based on the form of this new basis for the 5D calculus one can perform a corresponding rotation of transformation parameters,

$$\bar{\epsilon}_0 = (\hat{\epsilon}_0 + \hat{\epsilon}_4)/\sqrt{2}, \qquad \bar{\epsilon}_i \equiv \hat{\epsilon}_i, \qquad \bar{\epsilon}_4 = (\hat{\epsilon}_0 - \hat{\epsilon}_4)/\sqrt{2}, \tag{38}$$

and introduce the operators

$$\bar{\mathcal{D}}_0 = (\hat{P}_0 + \hat{P}_4)/\sqrt{2} = \mathcal{D}_0/\sqrt{2}, \qquad \bar{\mathcal{D}}_i \equiv \hat{P}_i, \qquad \bar{\mathcal{D}}_4 = (\hat{P}_0 - \hat{P}_4)/\sqrt{2}\,, \tag{39}$$

so that it is then possible to rewrite the $\hat{d}f$ we used in the previous sections in the following way

$$\hat{d}f \;=\; i\,(\bar{\epsilon}_0 \bar{\mathcal{D}}_0 + \bar{\epsilon}_i \bar{\mathcal{D}}_i + \bar{\epsilon}_4 \bar{\mathcal{D}}_4) f \equiv \bar{d}f\,. \tag{40}$$

It is clear from the structure of this redefinitions that by this way to rewrite $\hat{d}f$, while allowing to introduce a parameter ($\bar{\epsilon}_0$) that can be meaningfully described as a time-translation parameter, one does not affect in any "armful way" the progress of the Noether analysis. Indeed following the same steps we described in Section III, one easily arrives at the following expression for the Lagrangian-density variation

$$\delta\mathcal{L} = i\,\bar{\epsilon}^A \left(\bar{\mathcal{D}}^0 \bar{J}_{0A} + \bar{\mathcal{D}}^i \bar{J}_{iA}\right)\,, \tag{41}$$

where the quantities $\bar{J}_{\mu A}$ can be written in terms of the $J_{\mu A}$ (for which we gave the explicit functional dependence on the field in Section III) in the following simple way:

$$\bar{J}_{00} \equiv J_{00} + J_{04}, \qquad \bar{J}_{0i} \equiv \sqrt{2}\, J_{0i}, \qquad \bar{J}_{04} \equiv J_{00} - J_{04}\,,$$

$$\bar{J}_{i0} \equiv (J_{i0} + J_{i4})/\sqrt{2}, \qquad \bar{J}_{ik} \equiv J_{ik}, \qquad \bar{J}_{i4} \equiv (J_{i0} - J_{i4})/\sqrt{2}\,. \tag{42}$$

[7] Our core proposal here is that an operator proportional to $\mathcal{D}_0$ (or even just monotonic function of $\mathcal{D}_0$) should intervene explicitly in the "df rule". This can be done in many ways, but among these possibilities we chose to illustrate our idea in the case where the new basis is obtained from the $\hat{d}x_A$ basis by action of a rotation matrix of determinant 1. And it must be stressed that most of the welcome features of our new basis could be already achieved by essentially simply noticing that $\hat{\epsilon}^A\,\mathcal{D}^0\,J_{0A} = \mathcal{D}^0\big(\hat{\epsilon}^0(J_{00}+J_{04}) + \hat{\epsilon}^i J_{0i} + (\hat{\epsilon}^4 - \hat{\epsilon}^0)J_{04}\big)$.

[8] Actually Sitarz, in Ref. [19], already noticed that this change of basis $\hat{d}x_A \to \bar{d}x_A$ leads to simple commutation relations, but he had not realized that it would also allow a more intuitive characterization of time translations.

And then (proceeding again in complete analogy with what done in Section III) one obtains by spatial integration of the $\bar{J}_{0A}$ some conserved quantities $\bar{Q}_A$ which are themselves simply related to the $\hat{Q}_A$ derived in Section III:

$$\begin{pmatrix} \bar{Q}_0 \\ \bar{Q}_i \\ \bar{Q}_4 \end{pmatrix} = -\frac{1}{2}\int d^4k \left|\tilde{\Phi}(k)\right|^2 \begin{pmatrix} \hat{k}_0 + \hat{k}_4 \\ \sqrt{2}\,\hat{k}_i \\ \hat{k}_0 - \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2)}{|-2\tilde{k_0}e^{\frac{\lambda}{2}k_0} + \lambda m^2|}\,\delta(C_\lambda(k) - m^2) = \begin{pmatrix} \hat{Q}_0 + \hat{Q}_4 \\ \sqrt{2}\,\hat{Q}_i \\ \hat{Q}_0 - \hat{Q}_4 \end{pmatrix} . \tag{43}$$

In spite of the simplicity of the formula giving $\bar{Q}_0$ from $\hat{Q}_0$ and $\hat{Q}_4$, from a conceptual perspective $\bar{Q}_0$ might turn out to be a valuable tool, since it is the conserved charge associated with the transformation parameter $\bar{\epsilon}_0$, and therefore (in light of the fact that in $\bar{d}f$ we have $\bar{\epsilon}_0$ multiplying $\bar{\mathcal{D}}_0$, which is a plausible time-translation generator) is a plausible candidate for the energy charge.

VI. SOME REMARKS ON RELATIVISTIC STRUCTURES

As mentioned in the Introduction a primary source of interest in κ-Minkowski comes from the intuition that theories in this spacetime should be subject to a new type of spacetime symmetries. While the fact that we are finally able to perform some Noether analyses should prove useful for clarifying this possibility, we feel that at present nothing definite can be said. A key point is that, both using the 4D differential calculus [16] and the 5D differential calculus, one obtains charges that are time independent but for which it is not obvious how one should introduce a prescription for measurement. This is particularly true for the analysis based on the standard basis for the 5D calculus: the analysis itself suggests that none of the transformation parameters is a time-translation parameter, and consequently none of the charges can be viewed as a charge conserved under time-translation symmetry. The change of basis for the 5D differential calculus which we proposed in the previous section does lead to a "candidate time-translation-symmetry charge" which at least is plausible, but several logical-consistency checks (starting indeed by asking "how should one measure $\bar{Q}_0$?") should be performed before any definite claim.

This point is also relevant for the "equivalence between a free κ-Minkowski field theory and a free relativistically invariant (non-local) field theory on classical Minkowski space-time" proposed in Ref. [18]. We believe that at present such an equivalence simply cannot be established. This can be seen in many ways (comparing structures which emerged in our analysis to structures available in ordinary commutative theories), and perhaps most clearly by considering that on the commutative-theory side of the "equivalence" proposed in Ref. [18] the energy observable is readily available while on the non-commutative-theory side (since Ref. [18] makes reference to the standard basis for the 5D calculus) there is the mentioned issue concerning the energy observable.

Together with the energy-observable issues several other structures must be better understood before formulating any definite statement on relativistic properties. One (of possibly many) point to consider concerns the difference between the factor $\delta(C_\lambda(k) - m^2)$ in our formulas and the factor $\delta(k_0^2 - \vec{k}^2 - m^2)$ in the corresponding formulas found for theories in classical commutative Minkowski spacetime. It is perhaps noteworthy that

$$\delta(C_\lambda(k) - m^2) = \delta\left((\frac{2}{\lambda}\sinh\frac{\lambda k_0}{2})^2 - |\vec{k}|^2 e^{\lambda k_0} - m^2 \right) =$$

$$= \frac{1}{2\sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}} (\delta(k_0 - k_0^+) + \delta(k_0 - k_0^-)) \ , \tag{44}$$

where

$$k_0^+ = \frac{1}{\lambda}\ln\left(\frac{1 + (\lambda m)^2/2 + \lambda\sqrt{m^2 + |\vec{k}|^2 + \lambda^2 m^4/4}}{1 - (\lambda|\vec{k}|)^2} \right) \tag{45}$$

$$k_0^- = \frac{1}{\lambda}\ln\left(\frac{1+(\lambda m)^2/2-\lambda\sqrt{m^2+|\vec{k}|^2+\lambda^2 m^4/4}}{1-(\lambda|\vec{k}|)^2}\right) , \tag{46}$$

and k_0^+ is real only for $|\vec{k}| < 1/\lambda$. But here a complex k_0^+ may well be admissible[9], since in our theory the criterion one should enforce, which is the one of a real and positive energy charge, is at present not manageable (we do not know how to describe the energy observable, so we cannot test its reality and positivity).

Another key indicator of the relativistic structure of a theory is the energy-momentum dispersion relation, and of course also this indicator will not be available until a robust description of the energy observable is discovered. We thought that it might be worth looking for possible striking invariant combinations of the charges we obtained, but we found nothing noteworthy. One could attempt to identify such a combination of charges by probing the interconnection between the charges through a regularized plane-wave $\Phi^{p.w.}(x)$ solution of the equation of motion, of the following form:

$$\Phi^{p.w.}(x) = \frac{1}{(2V\sqrt{m^2+|k|^2+\lambda^2 m^4/4})^{\frac{1}{2}}} e^{i\vec{k}\cdot\vec{x}} e^{-ik_0^+ x_0}, \tag{47}$$

where V is a spatial-volume normalization factor and k_0^+ is related to $|\vec{k}|$ by (45).

Our results attribute to this field the charges

$$\begin{aligned} \hat{Q}_0^{p.w.} &= \hat{k}_0\big|_{k_0^+,\vec{k}}\,, \\ \hat{Q}_i^{p.w.} &= \hat{k}_i\big|_{k_0^+,\vec{k}}\,, \\ \hat{Q}_4^{p.w.} &= \hat{k}_4\big|_{k_0^+,\vec{k}}\,. \end{aligned} \tag{48}$$

It is perhaps intriguing that

$$(\hat{Q}_0^{p.w.})^2 - (\hat{Q}_i^{p.w.})^2 = m^2 + (\hat{Q}_4^{p.w.})^2 = m^2\left(1+\frac{\lambda^2 m^2}{4}\right) \tag{49}$$

but this should be analyzed taking into consideration the fact that, in light of the observations we reported above on time translations, $\hat{Q}_0^{p.w.}$ clearly cannot be the energy carried by our regularized plane wave.

In light of the observations we reported in the previous section one might consider contemplating a role for the combination $\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}$ (which gives a "$\bar{Q}_0^{p.w.}$" for our regularized plane wave), but we could not find any good use for it. For example, we find that

$$\left(\hat{Q}_0^{p.w.} + \hat{Q}_4^{p.w.}\right)^2 - \left(\hat{Q}_i\right)^2 = \left(\frac{e^{\lambda k_0^+}-1}{\lambda}\right)^2 - \left(k_i e^{\lambda k_0^+}\right)^2 = \left(\lambda\hat{Q}_0^{p.w.} + \lambda\hat{Q}_4^{p.w.} + 1\right) m^2 . \tag{50}$$

VII. COMPARISON WITH A RECENT RELATED ANALYSIS

While we were in the final stages of preparation of this manuscript we became aware of the very recent Ref. [20], by the same authors of Ref. [18], which considers the same framework we

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[9] Note however that there appears to be no obstruction for implementing the restriction $|\vec{k}| < 1/\lambda$ on the Fourier parameters. In fact, in a recent proposal of description of quantum fields in κ-Minkowski [25] this restriction was analyzed in some detail, finding that, upon adopting a suitable inner-product for the Hilbert space, it does not constitute a source of incompleteness in the construction of the Hilbert space. Instead by allowing complex values of k_0^+, and therefore $|\vec{k}| > 1/\lambda$, Ref. [25] found that the inner product was no longer guaranteed to be positive definite.

analyzed here (and they already analyzed in Ref. [18]). While it is probably fair to say that there is absolutely no overlap between the analysis we are here reporting and the one in Ref. [18], there is a correspondence between at least some points of our analysis and some corresponding points of the analysis reported in Ref. [20]. But a large number of crucial differences remain, and we hope to contribute to future further studies of this framework by stressing these differences, while acknowledging the points in common.

One first point of contact is that, while the analysis in Ref. [18] was only rather vaguely inspired by the 5D-calculus setup (which never explicitly appeared in the analysis), Ref. [20] uses explicitly the 5D calculus to construct a df, with just the same perspective and the same results we reported in Section II.

We should also mention that, while Ref. [18] only obtained 4 conserved charges from the 5D-calculus setup, Ref. [20] reports 5 conserved charges from the 5D-calculus setup. This is qualitatively consistent with our identification of 5 conserved charges. However, at the quantitative level (comparing the structure of the charges rather than just their abundance) there are significant differences between our results and the ones of Ref. [20] (and with the ones of Ref. [18]), and it is not hard to understand how these differences have emerged. In fact, while our Noether analysis constructively led us to a "conservation equation" of the form $\mathcal{D}_0 J^0_A + \hat{P}_i J^i_A = 0$, Ref. [20] somehow arrives at a conservation equation of the type[10] $\hat{P}_\mu J^\mu_A = 0$.

We have been unable to identify the assumption or choice which could have caused the analysis of Ref. [20] to end up with a $\hat{P}_\mu J^\mu_A = 0$ pseudo-conservation equation. However, we are confident that any recipe of Noether analysis leading to "would-be-conservation laws" of the form $\hat{P}_\mu J^\mu_A = 0$ should be rejected since, in light of the mentioned inadequacy of $\hat{P}_0$ to describe time derivatives, these are simply unacceptable as conservation equations.

Besides this crucial difference there are clearly many other differences between the two analyses, but the exercise to fully translate the analysis reported in Ref. [20] into formulas that are meaningful within our formulation of the problem is rather complex. One obstruction is caused by the fact that, while the equation of motion for scalar fields that we considered is the most studied such equation in the κ-Minkowski literature, Ref. [20] performs a symmetry analysis for a somewhat different, less known, equation of motion for scalar fields, which had been previously proposed by one of the authors. And a further difficulty is introduced by the fact that some of the formulas in Ref. [20] appear to be intended for quantum fields (although no Hilbert-space construction is offered), since they are described in terms of operators counting the number of particles; here instead we focused on the case of a Noether analysis of classical fields in our noncommutative spacetime.

VIII. SUMMARY AND OUTLOOK

While in Ref. [16], where some of us reported a first example of successful Noether analysis of Hopf-algebra spacetime symmetries, the description of translation transformations in 4D κ-Minkowski spacetime was based on the properties of a 4D differential calculus, more recently, in Ref. [18], it had been argued that there should also be a description of translations in 4D κ-Minkowski spacetime inspired by a 5D differential calculus. And while the derivation reported in Ref. [16] required some rather tedious manipulations of noncommutative functions and operator coproducts, the Noether analysis reported in Ref. [18] relied on a proposed equivalence between a free κ-Minkowski field theory and a free relativistically invariant (non-local) field theory on classical Minkowski space-time. We here exposed some limitations to the applicability of the proposed equivalence of theories, which in particular, as we showed, associates the operator of

[10] We have chosen, as a way to render the discussion clearer for our readers, to use our notation consistently throughout, even when reporting equations from Ref. [20]. For a direct comparison with what written in Ref. [20] it should be noticed that our $\hat{P}_\mu$ is denoted by $\hat{\partial}_\mu$ in Ref. [20], while for our $\mathcal{D}_0$ there is no dedicated symbol in Ref. [20] (our $\mathcal{D}_0$ would be described within the conventions adopted in Ref. [20] as $\hat{\partial}_0 + \hat{\partial}_4$).

time derivation on the commutative-theory side to an operator which does not even vanish on time-independent functions on the noncommutative-theory side.

However we also provided here further evidence in support of the possibility that the 5D-calculus-based "translation transformations" can indeed be implemented as symmetries of theories in κ-Minkowski. Our analysis performed directly within the noncommutative theory also allowed us to investigate explicitly the properties of the 5 "would-be currents" that one naturally ends up considering when working with the 5D calculus.

The fact that the techniques developed by some of us in Ref. [16] for a 4D-calculus-based description of translations were here successful, without any need of adaptation, in dealing with the very different 5D-calculus-based description of translations certainly encourages the hope that these techniques may be robust enough to deal with any kind of Hopf-algebra spacetime symmetries. We believe that a particularly striking indicator of the robustness of these techniques is provided by the fact that they automatically fixed an apparent problem of the standard basis for the 5D differential calculus, which leads to a 0-label generator acting in a way that would be inacceptable for a time-translation generator. Our approach constructively led us to current-conservation-like equations written in terms of the operator $\mathcal{D}_0$ which instead is a plausible candidate for the generation of time translations.

Concerning the puzzling apparent availability of different descriptions of translations transformations in κ-Minkowski our analysis did not lead to a definite answer, but, just because we showed that the description of the energy observable must be rather "tricky" within the 5D-calculus-based setup, it is legitimate to be hopeful: it is plausible that once we will have a robust understanding of the energy-momentum observables both in the 4D-calculus-based description and in the 5D-calculus-based description these two descriptions of translation transformations may turn out to be equivalent. The change of basis which we proposed in Section V may well turn out to be useful for this task.

The challenge of a proper identification of energy-momentum observables is also a necessary first step toward addressing the most significant issue here of interest, which concerns the fate of physical/observable aspects of spacetime symetries in noncommutative geometry. For this, besides the energy-momentum observables, one would also need to address other issues, some of which were preliminarily considered in our Section VI.

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